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Characterization for Best Nonlinear Approximations: A Geometrical Interpretation

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The present paper deals with several characterization theorems for best approximation in normed vector spaces by nonlinear elements. Guided by the outstanding results of Singer in the linear theory, some results of Laurent and Brosowski are generalized so as to obtain a unified approach for the linear and nonlinear approximation theory. Characterization theorems are formulated which assert the existence of particular linear functionals. We give geometrical interpretations to all our characterization theorems; also duality relations are given.

1. INTRODUCTION

Recently Singer presented a complete unified theory of approximation in a general normed linear space by elements of a linear subspace [10]. These results provide a modern theory of best approximation, which uses in a systematic manner the methods of functional analysis, general topology and geometry. Linear functionals play a central role in Singer's approach and this is mainly due to the duality relations between a given extremal problem in a linear space and the corresponding extremal problem in the dual space. Once the problem is embedded in this general context, proofs often become straightforward.

Brosowski introduced in [3] generalizations of the Kolmogoroff conditions, for nonlinear approximations in general normed linear spaces. To extend completely Singer's unified theory, a main theorem was still missing, asserting the existence of linear functionals with certain properties. In this contribution such a necessary and sufficient a condition is given, which shows the existence of some particular linear functionals (Lemma 7.). The necessary condition reduces in a particular case to the condition given by Laurent in [7], and earlier by Brosowoski in [2, p. 47] (restricted to a Chebyshev norm). However our deductions are independent of these results.

We give also a refined version of the above characterization, based on Singer's extension of Caratheodory's theorem. Effort is made to present results in a form similar to the unified approach of Singer. Consequently, differences between the linear and the nonlinear theory become apparent.

In Section 2 the general approximation problem is stated together with a list of some relevant concepts used throughout this paper. Most of them are standard [6]. In Section 3, the extensions of the Kolmogoroff conditions are reformulated and geometrically interpreted. They consist of a local necessary and a global sufficient condition and are called here characterizations of type I. Section 4 is devoted to a necessary and sufficient condition concerning the existence of particular linear functionals. A complete characterization theorem is obtained and called of type II (Theorem 8). If, in addition, a Gateaux (resp. Frechet) derivative exists for the approximating functions, this characterization theorem can be reformulated (Theorem 11). Geometrical interpretations of these characterization theorems are obtained. In Section 5 a refinement is given of the characterization theorems of type II which is, too, geometrically interpreted. Finally duality relations are obtained in Section 6.

2. Statement of the Problem. Notations

(2a) The problem of best approximation consists in finding, for a given function $f \in E$ (a normed linear space), an approximating function g_0 belonging to a given nonvoid subset G of E, such that:

$$\|f - g_0\| = \inf_{g \in G} \|f - g\|.$$

The set of all best approximating functions $g_0 \in G$ for f will be denoted by $\mathfrak{L}_G(f)$:

$$\mathfrak{L}_G(f) = \{g_0 \in G \mid \|f - g_0\| = \inf_{g \in G} \|f - g\|\}.$$

To exclude trivial cases we suppose G is not dense in E and $f \in E \backslash \operatorname{adh} G$.

(2b) It is convenient for nonlinear approximation problems to define a particular subset of the normed linear space E : [3, p. 147; 8, p. 2].

An element $h \in E$ will be called an *adherent displacement* for G starting from $g_0 \in G$ if for every neighborhood of h (denoted $\mathbf{N}_h$) and for every $\epsilon > 0$ there exists an $\eta \in]\, 0, \epsilon\, [$ and an $h' \in \mathbf{N}_h$ such that $g_0 + \eta \cdot h' \in G$.

The set of all adherent displacements will be denoted by $C[g_0, G]$; it is a nonvoid closed cone with vertex at the origin. ($h \in C \Rightarrow \lambda h \in C, \lambda > 0$).

By [8, p. 10] : if G is a convex subset of E and $g_0 \in \text{adh } G$, then the cone of adherent displacements $C[g_0, G]$ is also convex and is given by

$$C[g_0, G] = \text{adh } \left\{ \bigcup_{\lambda>0} \lambda(\text{adh } G - g_0) \right\}. \tag{1}$$

In general, the following inclusions hold:

$$C[g_0, G] \subset \text{adh } \left\{ \bigcup_{\lambda>0} \lambda(\text{adh } G - g_0) \right\} \supset \text{adh } G - g_0 .$$

In the particular case in which G is a linear subspace of E, the following identity is valid:

$$C[g_0, G] = \text{adh } G - g_0 .$$

(2c) Linear functionals play a central role in characterizing the best approximating function. We mention in this connection several concepts and properties.

Let E^* denote the *conjugate space* of the normed linear space E, namely, the space of all continuous linear functionals on E, endowed with the classical vector operations and the norm

$$\| L \| = \sup_{f \in B_E} | L(f) |, \qquad L \in E^*,$$

where $B_E = \{f \in E \mid \|f\| \leqslant 1\}$ denotes the *unit ball* in E. The space E^* will be provided with the *weak* topology* $\sigma(E^*, E)$ (simple convergence topology on E^*). The unit ball in E^*, denoted (B_{E^*}), is known to be compact for $\sigma(E^*, E)$ (theorem of Alaoglu). A set $\mathfrak{M}$ in a topological linear space is called an *extremal subset* of a closed convex set A, if $\mathfrak{M}$ is a nonvoid closed convex subset of A, and if the relations $x, y \in A$ and $\lambda x + (1 - \lambda) y \in \mathfrak{M}$, with $\lambda \in]0, 1[$, imply $x, y \in \mathfrak{M}$. An extremal subset of A consisting of a single point is called an *extremal point* of A. The set of all extremal points of A is denoted by $\mathfrak{E}(A)$. The set

$$\mathfrak{M}_f = \{L \in S_{E^*} \mid L(f) = \|f\|\}, f \in E \backslash \{0\}$$

(where $S_{E^*} = \{L \in E^* \mid \| L \| = 1\}$ is the unit sphere in E^*) is a nonvoid extremal subset of the ball B_{E^*} endowed with $\sigma(E^*, E)$ [10, p. 59], and is hence $\sigma(E^*, E)$-closed. Moreover, since B_{E^*} is compact in $\sigma(E^*, E)$, so is $\mathfrak{M}_f$. By [6, p. 78], $\mathfrak{E}(\mathfrak{M}_f)$ is nonvoid, and is $\mathfrak{E}(B_{E^*}) \cap \mathfrak{M}_f$ [10, p. 58].

The annihilator in E^* of a nonvoid subset A of the linear space E is

$$A^0 = \{L \in E^* \mid \quad L(y) = 0, \quad \forall y \in A \subset E\};$$

the annihilator in E of a nonvoid subset B of the linear space E^* will be denoted ${}^0B$. The number $\|f\|_\Gamma$, where $f \in E$ and $\Gamma \subset E^*$ is defined as

$$\|f\|_\Gamma = \sup_{L \in \Gamma \cap B_{E^*}} |L(f)|.$$

The restriction of a linear functional $L \in E^*$ to ${}^0\Gamma \subset E$ will be denoted $L|_{{}^0\Gamma}$.

(2d) In order to state clearly the geometrical interpretation of the characterizations, we need to introduce some geometrical concepts.

$$B(f, r) = \{g \in E \mid \|g - f\| \leqslant r\}$$

is a *ball* in the normed linear space E, and $H[L, \alpha] = \{g \in E \mid L(g) = \alpha\}$ is a *hyperplane* in E. The distance of an element $f \in E$ from the hyperplane $H[L, \alpha]$ is given by

$$\rho(f, H[L, \alpha]) = |L(f) - \alpha|/\|L\|.$$

The set $A \subset E$ supports the ball $B(f, r)$ if and only if $\rho(A, B(f, r)) = 0$ and the set $[A \cap \text{int. } B(f, r)]$ is void. By [10, p. 25] this is equivalent with $\rho(f, A) = r$. A real hyperplane $H[\text{Re } L, \alpha]$ is called an *extremal hyperplane* if $L \in \mathfrak{E}(B_{E^*})$. Moreover, in a normed linear space E, a real hyperplane $H[\text{Re } L, \alpha]$ is said to separate the subset $A \subset E$ from the subset $B \subset E$, if A is contained in one of the real half-spaces $\{k \in E \mid \text{Re } L(k) \geqslant \alpha\}$ or

$$\{k \in E \mid \text{Re } L(k) \leqslant \alpha\},$$

and B in the other.

A hyperplane H is said to pass through the set $\mathbf{M}$ if $\mathbf{M} \subset H$. The following property will prove to be very useful.

Property A. [10, p. 25]

Let E be a normed linear space, $f \in E$, $r > 0$. Then for any $L \in E^*$ with $L \in S_{E^*}$ the hyperplane $H[L, L(f) - r]$ supports the ball $B(f, r)$, and for any support hyperplane H of the ball $B(f, r)$ there exists a unique $L \in E^*$ with $L \in S_{E^*}$ such that $H = H[L, L(f) - r]$.

The approximation problem can be reformulated in this geometrical context, and consists in finding a point $g_0 \in G$ such that its distance to f (denoted $\rho(f, g_0)$) equals the distance of f to G,

$$\rho(f, G) = \inf_{g \in G} \rho(f, g).$$

All points g_0 satisfying this requirement form the set $\mathfrak{L}_G(f)$. In linear approximation theory (G a linear subspace of E), $g_0 \in \mathfrak{L}_G(f)$ is equivalent to the existence of a linear functional $L \in \mathfrak{M}_{f-g_0} \cap G^0$, defining a hyperplane $H[L, 0]$, which passes through G and supports the ball $B(f, \|f - g_0\|)$.

Consequently, the linear space G and the hyperplance H are at equal distance to f.

In the following we will give the geometrical interpretation of the extension of these results.

3. Characterization Theorems of Type I

(3a) *Extended Kolmogoroff Conditions*

The following Lemma 1 and Lemma 2 are known extensions of the Kolmogoroff condition [3]. They are presented here in a form suitable for geometrical interpretation; see Singer [10, 59–62].

LEMMA 1. [3, p. 148]. *Let E be a normed linear space and G a subset of E, with $f \in E \backslash adh\ G$ and $g_0 \in G$. If $g_0 \in \mathfrak{L}_G(f)$, then for every $h \in C[g_0, G]$, there exists a linear functional $L^h \in E^*$ such that*:

(i) $L^h \in \mathfrak{E}(B_{E^*})$, (2)

(ii) $L^h(f - g_0) = \|f - g_0\|$, (3)

(iii) $\operatorname{Re} L^h(h) \leqslant 0.$ (4)

LEMMA 2. [3, p. 141]. *Let E be a normed linear space and G a subset of E, with $f \in E \backslash adh\ G$ and $g_0 \in G$. If for every $g \in G$ there exists a linear functional $L^g \in E^*$ such that*:

(i) $L^g \in \mathfrak{E}(B_{E^*})$, (5)

(ii) $L^g(f - g_0) = \|f - g_0\|$, (6)

(iii) $\operatorname{Re} L^g(g - g_0) \leqslant 0,$ (7)

then $g_0 \in \mathfrak{L}_G(f)$.

(i)–(iii) of Lemma 1 are known as the *Local Kolmogoroff condition on G*. Their *necessity* is presented in [3] and [5, p. 370] as : if $g_0 \in \mathfrak{L}_G(f)$, then for every $h \in C[g_0, G]$,

$$\min_{L \in \mathfrak{E}(\mathfrak{M}_{f-g_0})} \operatorname{Re} L(h) \leqslant 0. \tag{8}$$

Similarly (i)–(iii) of Lemma 2 are known as the *Global Kolmogoroff condition on G*. Their *sufficiency* can also be stated in the following form, according to [3] and [5, p. 370]: if for every element $g \in G$,

$$\min_{L \in \mathfrak{E}(\mathfrak{M}_{f-g_0})} \operatorname{Re} L(g - g_0) \leqslant 0, \tag{9}$$

then $g_0 \in \mathfrak{L}_G(f)$.

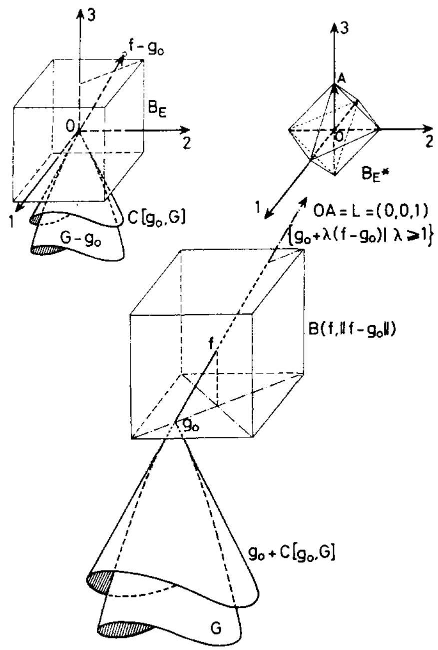

FIG. 1. The Local and Global Kolmogoroff conditions in R^3.

The Fig. 1 will be helpful in interpreting the preceding Lemmas. For the problem of approximating $f \in E$ by elements of $G \subset E$ where the normed space $E = R^3$, we obviously have $g_0 \in \mathfrak{L}_G(f)$. By Lemma 1 there exists a unique linear functional L satisfying $L \in \mathfrak{E}(B_{E^*}) \cap \mathfrak{M}_{f-g_0}$, where $L(h) = h_3$ for $\forall h \in R^3$, $h = (h_1, h_2, h_3)$. The local Kolmogoroff condition (2)–(4) or (8) is satisfied since $\operatorname{Re} L(h) \leqslant 0$ for $\forall h \in C[g_0, G]$. The Global Kolmogoroff condition on G is also satisfied for the unique L defined above. Since

$$\operatorname{Re} L(g') \leqslant 0 \text{ for } \forall g' \in G - g_0,$$

by (5)–(7) or (9), $g_0 \in \mathfrak{L}_G(f)$. By Fig. 1 it is easily verified that the Local Kolmogoroff condition is only necessary. Remembering that in general the subset G of E is only partially contained in $C[g_0, G]$, we see that even if (8) is satisfied for all elements $h \in C[g_0, G]$, there may exist an element $g \in G$ belonging to $\operatorname{int} B(f, \|f - g_0\|)$, so that $g_0 \notin \mathfrak{L}_G(f)$. Similarly, the Global Kolmogoroff condition is only sufficient. Indeed if $g_0 \in \mathfrak{L}_G(f)$, there may exist some $g \in G$ for which (9) is not satisfied.

The condition (2)–(4) of Lemma 1, as well as (5)–(7) of Lemma 2 can be expressed in equivalent forms. In particular, for the condition of Lemma 2 we have the following equivalent variants:

COROLLARY 3. *Let E be a normed linear space and G a subset of E, with* $f \in E \backslash \operatorname{adh} G$ *and* $g_0 \in G$. *If for every* $g \in G$ *there exists a linear functional* $L^g \in E^*$ *such that one of the following equivalent conditions is satisfied*:

(a) (i) $L^g \in \mathfrak{E}(\mathfrak{M}_{f-g_0})$,

(ii) $\operatorname{Re} L^g(g_0 - g) \geqslant 0.$ (10)

(b) (5), (10) *and* $\operatorname{Re} L^g(f - g_0) = \| f - g_0 \|$.

(c) (i) (5),

(ii) $\operatorname{Re} [L^g(g_0 - g) \cdot \overline{L^g(f - g_0)}] \geqslant 0$,

(iii) $| L^g(f - g_0)| = \| f - g_0 \|$,

then $g_0 \in \mathfrak{L}_G(f)$.

Proof. Obviously (a) → (b). We have (b) → (a) since

$$\| f - g_0 \| = \operatorname{Re} L^g(f - g_0) \leqslant | L^g(f - g_0)| \leqslant \| f - g_0 \|$$

and consequently $L^g(f - g_0)$ is real and $\geqslant 0$. Obviously we also have (a) → (c). To prove (c) → (a), we define the linear functional

$$\mathfrak{L}^g = [\operatorname{sign} L^g(f - g_0)] \cdot L^g, \text{ where } \operatorname{sign} \alpha = \bar{\alpha}/| \alpha |.$$

Consequently $\mathfrak{L}^g \in \mathfrak{E}(B_{E^*})$ and

$$\operatorname{Re} \mathfrak{L}^g(g_0 - g) = \operatorname{Re}[L^g(g_0 - g) \cdot \overline{L^g(f - g_0)}] \geqslant 0$$

$$\mathfrak{L}^g(f - g_0) = | L^g(f - g_0)| = \| f - g_0 \|.$$

Q.E.D.

Remark. In the particular case in which G is a linear subspace of E, the facts $C[g_0, G] = \operatorname{adh} G - g_0$ and $g - g_0 \in G$ for $\forall g \in G$, reduce Lemma 1 and Lemma 2, respectively, to the necessary and sufficient parts of the Kolmogoroff condition

$$\min_{L \in \mathfrak{E}(\mathfrak{M}_{f-g_0})} \operatorname{Re} L(g) \leqslant 0; \qquad \forall g \in G$$

for g_0 to satisfy $g_0 \in \mathfrak{L}_G(f)$ [10, p. 62]. The condition can also be stated as

$$\max_{L \in \mathfrak{E}(\mathfrak{M}_{f-g_0})} \operatorname{Re} L(g) \geqslant 0; \qquad \forall g \in G.$$

(3b) *Properties* (*B*)

(B1) If G is a nonvoid subset of E, the following statements are equivalent [5, p. 371, 383]:

(i) The Global Kolmogoroff condition on G is necessary,

(ii) For every $f \in E$, all elements $g \in G$ satisfying $g \in \mathfrak{L}_G(f)$ satisfy also $g \in \mathfrak{L}_G(g + \lambda(f - g))$ for all $\lambda \geqslant 1$.

(Fig. 1: $g_0 \in \mathfrak{L}_G(g_0 + \lambda(f - g_0)), \forall\lambda \geqslant 1$).

(B2) If the subset G of E satisfies $G \subset C[g_0, G] + g_0$, then:

(i) The Local Kolmogoroff condition on G is sufficient,

(ii) The Global Kolmogoroff condition on G is necessary.

Proof. If $g_0 \in \mathfrak{L}_G(f)$, then (8) is valid for $\forall h \in C[g_0, G]$ and consequently (8) is valid for $\forall h \in G - g_0$ which means that (9) is valid for $\forall g \in G$. Q.E.D.

(B3) If G is convex, then by (1), $C[g_0, G] \supset G - g_0$ and consequently (i) and (ii) of (B2) hold.

(B4) If $g_0 \in \mathfrak{L}_G(f)$ then:

(i) $0 \in \mathfrak{L}_{C[g_0, G]}(f - g_0)$,

(ii) $g_0 \in \mathfrak{L}_{C[g_0, G]+g_0}(f)$.

Proof. By the Local Kolomogoroff condition, (8) is satisfied for $\forall h \in C[g_0, G]$, which by the Global Kolmogoroff condition on $C[g_0, G]$ proves $0 \in \mathfrak{L}_{C[g_0, G]}(f - g_0)$. Q.E.D.

(B5) The following statements are equivalent

(i) $0 \in \mathfrak{L}_{C[g_0, G]}(f - g_0)$,

(ii) $\min_{L \in \mathfrak{E}(\mathfrak{M}_{f-g_0})} \operatorname{Re} L(h) \leqslant 0, \qquad \forall h \in C[g_0, G]$.

Proof. If $0 \in \mathfrak{L}_{C[g_0, G]}(f - g_0)$, then by the Local Kolmogoroff condition on $C[g_0, G]$ we have (8) for $\forall h \in C[0, C[g_0, G]]$. We have $C[g_0, G] = C[0, C[g_0, G]]$, since, applying the general inclusion for the cone of adherent displacements, we obtain $C[0, C[g_0, G]] \subset C[g_0, C]$. Conversely, applying the definition of the cone of adherent displacements, it becomes obvious that $y \in C[g_0, G]$ implies $y \in C[0, C[g_0, G]]$. Consequently we have (8) for $\forall h \in C[g_0, G]$.

If (ii) is satisfied, by the global Kolmogoroff sufficient condition we obtain immediately (i). Consequently, the Global Kolmogoroff condition on $C[g_0, G]$ is necessary and sufficient. Q.E.D.

(3c) *Geometrical Interpretation*

We deduce first a theorem expressing in geometrical terms, the requirements of the Kolmogoroff conditions on G.

THEOREM 4. *Let E be a normed linear space, G a subset of E, $f \in E \backslash \text{adh}\, G$ and $g_0 \in G$. Let $L \in B_{E^*}$ and let h be a given element of E. The following statements are equivalent.*

(a) *The linear functional $L \in E^*$ satisfies*

(i) $L \in \mathfrak{E}(B_{E^*})$,

(ii) $\text{Re}\, L(h) \leqslant 0$,

(iii) $L(f - g_0) = \|f - g_0\|$.

(b) *The real support hyperplane $H[\text{Re}\, L, \text{Re}\, L(f) - \|f - g_0\|]$ of the ball $B(f, \|f - g_0\|)$ is extremal, passes through g_0 and separates $\{h + g_0\}$ from $B(f, \|f - g_0\|)$.*

Proof. The real support hyperplane is extremal if and only if $L \in \mathfrak{E}(B_{E^*})$. If $L(f - g_0) = \|f - g_0\|$, it follows immediately that

$$H[\text{Re}\, L, \text{Re}\, L(f) - \|f - g_0\|]$$

passes through g_0. Conversely, if we have

$$\|f - g_0\| = \text{Re}\, L(f - g_0) \leqslant |L(f - g_0)| \leqslant \|f - g_0\|,$$

then $L(f - g_0)$ is real and $\geqslant 0$, and $L(f - g_0) = \|f - g_0\|$. For

$$\forall y \in B(f, \|f - g_0\|)$$

we have

$$\|f - g_0\| \geqslant |L(f - y)| \geqslant \text{Re}\, L(f - y)$$

and consequently for H to separate $\{h + g_0\}$ from $B(f, \|f - g_0\|)$ it is necessary and sufficient that

$$\text{Re}\, L(h + g_0) \leqslant \text{Re}\, L(f) - \|f - g_0\|.$$

Assuming that we have $L(f - g_0) = \|f - g_0\|$, or that H passes through g_0, we obtain $\text{Re}\, L(h) \leqslant 0$ for the element h of E. Q.E.D.

The geometrical interpretation of the Local Kolmogoroff condition is now easily obtained as

THEOREM 5. *Let E be a normed linear space, G a subset of E, $f \in E \backslash \text{adh}\, G$ and $C[g_0, G]$ a nonvoid subset of E. If $g_0 \in \mathfrak{L}_G(f)$, then for every $h \in C[g_0, G]$ there exists a real extremal hyperplane H^h which supports the ball*

$$B(f, \|f - g_0\|),$$

passes through g_0 and separates $\{h + g_0\}$ from $B(f, \|f - g_0\|)$.

If, in addition, the Local Kolmogoroff condition is also sufficient, then the preceding statement is equivalent with $g_0 \in \mathfrak{L}_G(f)$.

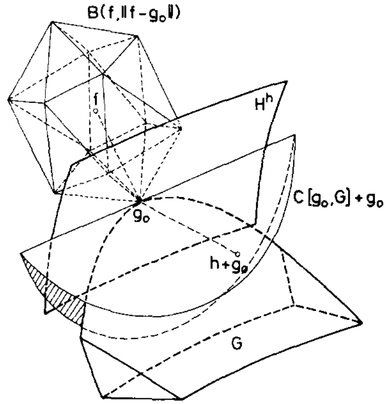

FIG. 2. Geometrical interpretation of the Local Kolmogoroff condition.

This geometrical interpretation of the Local Kolmogoroff condition is shown in Fig. 2.

Analogously, the Global Kolmogoroff condition can be interpreted as follows:

THEOREM 6. *Let E be a normed linear space, G a subset of E and $f \in E \backslash \text{adh}\, G$. If for every $g \in G$, there exists a real extremal hyperplane H^g which supports the ball $B(f, \|f - g_0\|)$, passes through g_0 and separates g from $B(f, \|f - g_0\|)$, then $g_0 \in \mathfrak{L}_G(f)$.*

If, in addition, the Global Kolmogoroff condition is necessary, then the preceding condition is equivalent with $g_0 \in \mathfrak{L}_G(f)$.

The geometrical interpretation of the Global Kolmogoroff condition is represented in Fig. 3.

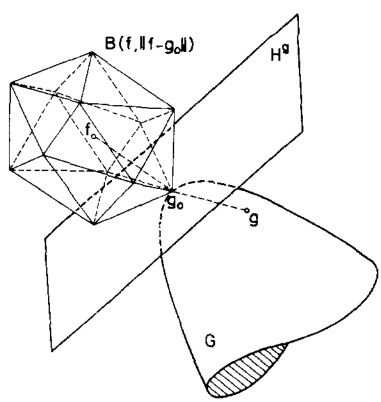

FIG. 3. Geometrical interpretation of the Global Kolmogoroff condition.

4. CHARACTERIZATION THEOREMS OF TYPE II

(4a) Based on the Hahn–Banach extension theorem, the existence of linear functionals with particular properties can be proved. They play a crucial role in nonlinear approximation theory, as in the linear case [10, p. 18]. We deduce first a general necessary and also a sufficient condition for the existence of these functionals.

LEMMA 7. *Let E be a normed linear space and G a subset of E, $f \in E \backslash \text{adh}\, G$ and $g_0 \in G$.*

(a) If (i) $g_0 \in \mathfrak{L}_G(f)$,

(ii) $\mathbf{M}$ *is a nonvoid linear subspace in E (not necessarily closed) such that* $0 \in \mathfrak{L}_{\mathbf{M}}(f - g_0)$, *then* $\mathfrak{M}_{f-g_0} \cap \mathbf{M}^0$ *is a nonvoid subset of* E^*.

(b) *If* (i) *the Local Kolmogoroff condition on G is sufficient*

(ii) *the set* $\mathfrak{M}_{f-g_0} \cap C^0[g_0, G]$ *is a nonvoid subset of* E^*, *then* $g_0 \in \mathfrak{L}_G(f)$.

Proof. (a) Since $f \in E \backslash \text{adh}\, G$ and $g_0 \in \mathfrak{L}_G(f)$, we have $\|f - g_0\| > 0$. This together with $0 \in \mathfrak{L}_{\mathbf{M}}(f - g_0)$ imply $f - g_0 \in E \backslash \text{adh}\, \mathbf{M}$. Applying Singer's theorem [10, Theorem 1.1, p. 18] to the linear approximation of $(f - g_0)$ by elements of the linear subspace $\mathbf{M}$ of E, we have that $0 \in \mathfrak{L}_{\mathbf{M}}(f - g_0)$, $0 \in \mathbf{M}$, is equivalent with the existence of a linear functional L in E^* such that $\|L\| = 1$, $L(h) = 0$ for $\forall h \in \mathbf{M}$ and $L(f - g_0) = \|f - g_0\|$.

(b) If the linear functional $L \in E^*$ satisfies $L \in \mathfrak{M}_{f-g_0} \cap C^0[g_0, G]$, we have

$$\|f - g_0\| = L(f - g_0 - k) \leqslant \|f - g_0 - k\| \text{ for } \forall k \in C[g_0, G]$$

and consequently $0 \in \mathfrak{L}_{C[g_0, G]}(f - g_0)$. By the property (B5) we obtain (8) for all $h \in C[g_0, G]$. Finally by the fact that the Local Kolmogoroff condition on G is sufficient, we have $g_0 \in \mathfrak{L}_G(f)$. Q.E.D.

Remarks (R).

(R1) In some particular approximation problems, Lemma 7(b) can be reformulated. We obtain the following corollaries by the properties (B2) and (B3).

COROLLARY 7(c). *Let E be a normed linear space, G a subset of E satisfying* $G \subset C[g_0, G] + g_0$, $g_0 \in G$, $f \in E \backslash \text{adh}\, G$. *If* $\mathfrak{M}_{f-g_0} \cap C^0[g_0, G]$ *is a nonvoid subset of* E^*, *then* $g_0 \in \mathfrak{L}_G(f)$.

COROLLARY 7(d). *Let E be a normed linear space, G a convex subset of E,* $f \in E \backslash \text{adh}\, G$ *and* $g_0 \in G$. *If* $\mathfrak{M}_{f-g_0} \cap C^0[g_0, G]$ *is a nonvoid subset of* E^*, *then* $g_0 \in \mathfrak{L}_G(f)$.

(R2) If G is a linear subspace in E, then Lemma 7(a) and Corollary 7(d) reduce to Singer's Theorem 1.1 [10, p. 18] on the equivalence between $g_0 \in \mathfrak{L}_G(f)$ and the nonvoidness of $(\mathfrak{M}_{f-g_0} \cap G^0)$.

Lemma 7(a) and Corollary 7(d) are respectively, represented in Fig. 4 and Fig. 5 for the particular case $E = R^3$.

By Lemma 7 it becomes obvious that a characterization theorem of type II will be obtained if the cone $C[g_0, G]$ could be replaced by a nonvoid linear subspace of $C[g_0, G]$. These requirements are very restricting and consequently are not fulfilled in general: there is no guarantee for $C[g_0, G]$ to contain a nonvoid linear subspace. In general, the cone $C[g_0, G]$ will contain a line through the origin if with a given function $h \in C[g_0, G]$, the function $(-h)$ is also contained in $C[g_0, G]$. If the cone $C[g_0, G]$ with vertex at the origin is convex (which is equivalent with $C[g_0, G] + C[g_0, G] \subset C[g_0, G]$ and $\lambda \cdot C[g_0, G] \subset C[g_0, G]$ for all $\lambda > 0$), then the largest linear subspace contained in the cone $C[g_0, G]$ is given by

$$C[g_0, G] \cap (-C[g_0, G]) \qquad [1, \text{p. } 47].$$

In the following we will formulate a characterization theorem, supposing $C[g_0, G]$ contains at least a nonvoid linear subspace $\mathbf{M}_c[g_0, G]$. (i.e., a line through the origin). In order to replace the cone of adherent displacements $C[g_0, G]$ by $\mathbf{M}_c[g_0, G]$ we introduce a *Local Kolmogoroff condition on G versus* $\mathbf{M}_c[g_0, G]$: if $g_0 \in \mathfrak{L}_G(f)$, then for every $h \in \mathbf{M}_c[g_0, G]$,

$$\min_{L \in \mathfrak{E}(\mathfrak{M}_{f-g_0})} \operatorname{Re} L(h) \leqslant 0. \tag{11}$$

This condition is always necessary for $g_0 \in \mathfrak{L}_G(f)$.

Properties (C)

(C1) If the Local Kolmogoroff condition on G versus $\mathbf{M}_c[g_0, G]$ is sufficient, then the Local Kolmogoroff condition on G is also sufficient.

Proof. If there exists an element $h \in C[g_0, G] \backslash \mathbf{M}_c[g_0, G]$ such that (11) is not satisfied, then the Local Kolmogoroff condition on G is contradicted. QED

(C2) Let E be a normed linear space, G a subset of E, $f \in E \backslash \operatorname{adh} G$, $g_0 \in G$, and $\mathbf{M}_c[g_0, G]$ a nonvoid linear subspace of the cone $C[g_0, G]$ ($\mathbf{M}_c[g_0, G]$ is not necessarily closed).

If $g_0 \in \mathfrak{L}_G(f)$, then $0 \in \mathfrak{L}_{\mathbf{M}_c[g_0, G]}(f - g_0)$, or $g_0 \in \mathfrak{L}_{\mathbf{M}_c[g_0, G] + g_0}(f)$.

Proof. If we have $g_0 \in \mathfrak{L}_G(f)$, then the Local Kolmogoroff condition is always necessary. Consequently, $\forall h \in C[g_0, G]$,

$$\min_{L \in \mathfrak{E}(\mathfrak{M}_{f-g_0})} \operatorname{Re} L(h) \leqslant 0$$

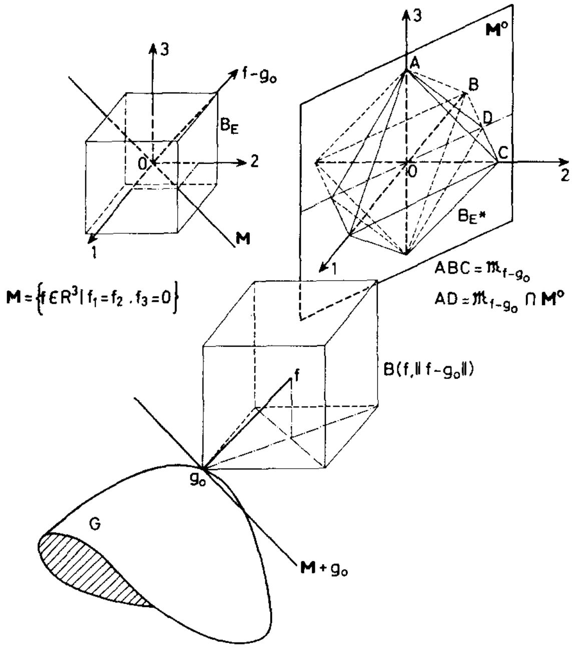

FIG. 4. If $g_0 \in \mathfrak{L}_G(f)$ then set $\mathfrak{M}_{f-g_0} \cap \mathbf{M}^0$ is nonvoid.

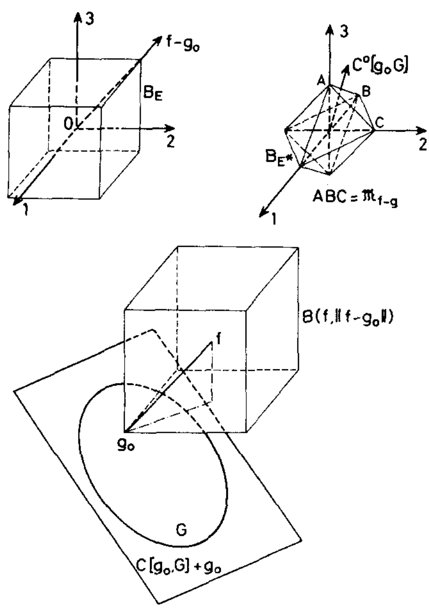

FIG. 5. If $\mathfrak{M}_{f-g_0} \cap C^0[g_0, G]$ is nonvoid in E^* and G convex then $g_0 \in \mathfrak{L}_G(f)$.

and hence

$$\min_{L \in \mathfrak{E}(\mathfrak{M}_{f-g_0}-0)} \operatorname{Re} L(h-0) \leqslant 0, \qquad \forall h \in \mathbf{M}_c[g_0, G]$$

which by the Global Kolmogoroff condition is necessary and sufficient for $0 \in \mathbf{M}_c[g_0, G]$ to satisfy

$$0 \in \mathfrak{L}_{\mathbf{M}_c[g_0, G]}(f - g_0).$$

Q.E.D.

Remark. Since $\mathbf{M}_c[g_0, G]$ is a linear subspace, the Global Kolmogoroff condition is always necessary and sufficient for $0 \in \mathfrak{L}_{\mathbf{M}_c[g_0, G]}(f - g_0)$.

We obtain now a complete characterization theorem of type II involving the existence of special linear functionals in E^*.

THEOREM 8. *Let E be a normed linear space and G a subset of E, let $f \in E \backslash \operatorname{adh} G$ and $g_0 \in G$. Let $\mathbf{M}_c[g_0, G]$ be a nonvoid linear subspace of E contained in the cone $C[g_0, G]$.*

(a) *If $g_0 \in \mathfrak{L}_G(f)$, then $\mathfrak{M}_{f-g_0} \cap \mathbf{M}_c^0[g_0, G]$ is nonvoid.*

(b) *If the Local Kolmogoroff condition on G versus $\mathbf{M}_c[g_0, G]$ is sufficient, then $g_0 \in \mathfrak{L}_G(f)$ if and only if $\mathfrak{M}_{f-g_0} \cap \mathbf{M}_c^0[g_0, G]$ is nonvoid.*

Proof. The necessity of the condition follows immediately from Property (C2) and Lemma 7(a). We only need to prove sufficiency. If the linear functional $L \in E^*$ satisfies $L \in \mathfrak{M}_{f-g_0} \cap \mathbf{M}_c^0[g_0, G]$, then

$$\| f - g_0 \| = L(f - g_0 - k) \leqslant \| f - g_0 - k \| \text{ for } \forall k \in C[g_0, G]$$

and consequently: $0 \in \mathfrak{L}_{\mathbf{M}_c[g_0, G]}(f - g_0)$.

Since $\mathbf{M}_c[g_0, G]$ is a linear subspace of E, we obtain (11) for $\forall h \in \mathbf{M}_c[g_0, G]$ which is equivalent with $g_0 \in \mathfrak{L}_G(f)$ since the Local Kolmogoroff condition on G versus $\mathbf{M}_c[g_0, G]$ is supposed to be sufficient. Q.E.D.

In the particular case in which G is a linear subspace of E, Theorem 8 reduces to Singer's Theorem 1.1 [10, p. 18] with $\mathbf{M}_c[g_0, G] = G$. According to Singer, Theorem 8 can be formulated in some equivalent forms.

COROLLARY 9 [10, p. 19; Lemma 1.1]. *Let E be a normed linear space, G a subset of E, $f \in E \backslash \operatorname{adh} G$ and $g_0 \in G$. Let $\mathbf{M}$ be a linear subspace in E. The following statements a, b, c, d, e and f*

(a) $L \in \mathfrak{M}_{f-g_0} \cap \mathbf{M}^0$.

(b) (i) $L \in S_{E^*} \cap \mathbf{M}^0$, (12)

(ii) $\operatorname{Re} L(f - g_0) = \| f - g_0 \|$. (13)

(c) (i) $L \in S_{E^*}$, (14)

(ii) $\operatorname{Re} L(h) = 0, \forall h \in \mathbf{M}$, (15)

(iii) (13).

(d) (i) (12),

(ii) $| L(f - g_0)| = \| f - g_0 \|$.

(e) (i) (12),

(ii) $| \operatorname{Re} L(f - g_0)| = \| f - g_0 \|$. (16)

(f) (14), (15) and (16).

(4b) *Particular Linear Subspaces Contained in* $C[g_0, G]$

Using the general concept of differentiation [9, Chapter 3] we give explicit examples of linear spaces $\mathbf{M}_c[g_0, G]$ contained in the cone $C[g_0, G]$. Suppose $G \subset E$ satisfies $G = \{g(a) | a \in P\}$ where P is an open subset in a normed vector space $\mathfrak{E}$.

(4b)(1) Assume g to be *Gateaux differentiable* at an interior point a of P, which means that there exists a linear operator $A \in \mathfrak{L}[\mathfrak{E}, E]$ such that for any $b \in \mathfrak{E}$:

$$\lim_{t \to 0} \left(\frac{1}{t}\right) \cdot \| g(a + tb) - g(a) - t \cdot Ab \| = 0. \tag{17}$$

The unique linear operator $A \in \mathfrak{L}[\mathfrak{E}, E]$ for which (17) holds, will be denoted by $g_G'(a)$ and called the *Gateaux derivative* of g at a. Consider now the linear subspace $G[a]$ of E, defined as

$$G[a] = \{Ab \mid A = g_G'(a), b \in \mathfrak{E}\}.$$

Then $G[a] \subset C[g(a), G]$ since for any given positive numbers δ and ϵ, as g is Gateaux differentiable, there exists a $t_0 > 0$ such that $t_0 < \epsilon$ and

$$\left\| Ab - \frac{g(a + t_0 b) - g(a)}{t_0} \right\| < \delta.$$

Taking $\eta = t_0$, one obtains

$$g(a) + \eta \cdot \frac{g(a + t_0 b) - g(a)}{t_0} \in G \subset E.$$

(4b)(2) We consider now a stronger form of differentiation. The mapping g is *Frechet differentiable* at a point $a \in P$ if there is a linear operator $\mathfrak{A} \in \mathfrak{L}[\mathfrak{E}, E]$ such that

$$\lim_{b \to 0} \frac{1}{\| b \|_{\mathfrak{E}}} \cdot \| g(a + b) - g(a) - \mathfrak{A} b \| = 0.$$

The linear operator $\mathfrak{A}$ is denoted by $g_{\mathfrak{F}}'(a)$ and called the *Frechet derivative* of g at a. Defining the linear subspace in E,

$$\mathfrak{F}[a] = \{\mathfrak{A}b \mid \forall b \in \mathfrak{E},\ \mathfrak{A} = g_{\mathfrak{F}}'(a)\},$$

we have $\mathfrak{F}[a] \subset C[g(a), G]$

(4b)(3). The mapping g is said to have a *Gateaux differential* at a in the direction b, if the limit

$$\lim_{t\to 0} \left(\frac{1}{t}\right) \cdot (g(a + tb) - g(a)) = V(a, b)$$

exists.

If $V(a, b)$ exists for every $b \in \mathfrak{E}$ and if $V(a, b)$ is linear in b (which means $V(a, b) = A(a) \cdot b$; $A(a) \in \mathfrak{L}(\mathfrak{E}, E)$), then $A(a)$ is the *Gateaux derivative* of g at a; $A(a) = g'(a)$. If the Gateaux differential at a exists for all $b \in \mathfrak{E}$ and if

$$\lim_{b\to 0} \frac{1}{\| b \|} \cdot (\| g(a + b) - g(a) - V(a, b)\|) = 0, \qquad (18)$$

then g has a *Frechet differential* at a. Denoting

$$F[a] = \{V(a, b) \mid b \in \mathfrak{E}\}$$

we have $F[a] \subset C[g(a), G]$; but $F[a]$ is not a linear space. In the following $\mathfrak{L}[a_0]$ will stand for $G[a_0]$ or $\mathfrak{F}[a_0]$, if they are nonvoid; it is a particular linear subspace of $C[g_0, G]$.

COROLLARY 10. *Let E be a normed linear space and let $G = \{g(a) \mid a \in P\}$ be subset of E such that for $\forall a \in P$, $\mathfrak{L}[a]$ is nonvoid, $f \in E\backslash$adh G and $g_0 \in G$. If $g_0 \in \mathfrak{L}_G(f)$, then $\mathfrak{M}_{f-g_0} \cap \mathfrak{L}^0[a_0]$ is nonvoid.*

THEOREM 11. *Let E be a normed linear space and G subset of E such that for $\forall a \in P$, $\mathfrak{L}[a]$ is nonvoid. Let $f \in E\backslash$adh G and $g_0 \in G$. If the Local Kolmogoroff condition on G versus $\mathfrak{L}[a_0]$ is sufficient, then $g_0 \in \mathfrak{L}_G(f)$ if and only if $\mathfrak{M}_{f-g_0} \cap \mathfrak{L}^0[a_0]$ is nonvoid.*

In Fig. 6 and Fig. 7 an example is given for the set $\mathfrak{M}_{f-g_0} \cap \mathfrak{L}^0[a_0]$, corresponding to a particular approximation problem in R^3.

Remark. Corollary 10 was given by Laurent in [7, p. 247; Theorem 2] for the particular case $\mathfrak{L}[a] = G[a]$.

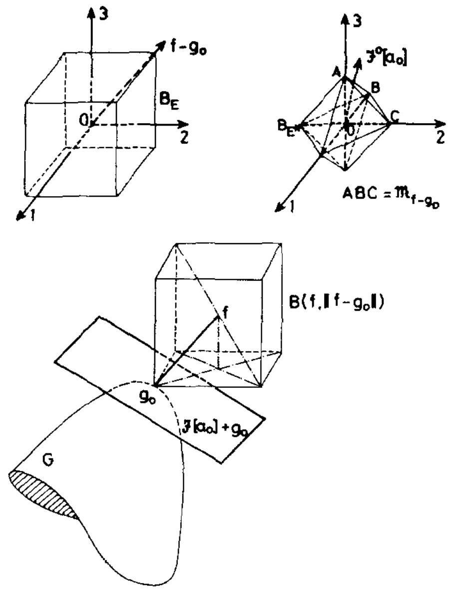

FIG. 6. If $g_0 \in \mathfrak{L}_G(f)$ then $\mathfrak{M}_{f-g_0} \cap \mathfrak{F}^0[a_0]$ is nonvoid.

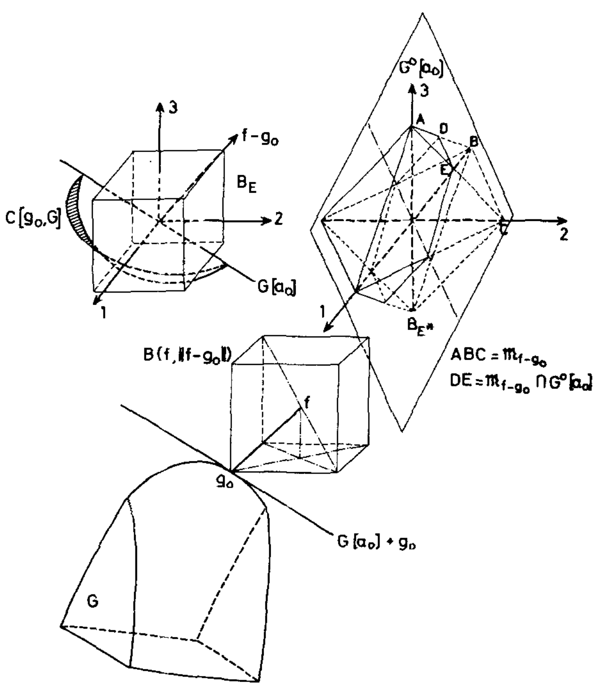

FIG. 7. If $g_0 \in \mathfrak{L}_G(f)$ then $\mathfrak{M}_{f-g_0} \cap G^0[a_0]$ is nonvoid.

(4c) *Geometrical Interpretation.*

To obtain a geometrical interpretation of the characterization theorem of type II, we deduce first a theorem which will interpret the nonvoidness of $\mathfrak{M}_{f-g_0} \cap \mathbf{M}_c{}^0[g_0, G]$ and in particular of $\mathfrak{M}_{f-g_0} \cap \mathfrak{L}^0[a_0]$.

THEOREM 12. *Let E be a normed linear space, G a subset of E, let $f \in E \backslash \operatorname{adh} G$ and $g_0 \in G$. Let A be a nonvoid subset of E containing 0. The following statements are equivalent:*

(1) *The set $\mathfrak{M}_{f-g_0} \cap A^0 \subset E^*$ is nonvoid.*

(2) *The hyperplane $H[L, L(f) - \|f - g_0\|]$ supports the ball*

$$B(f, \|f - g_0\|)$$

and passes through the translated set $(A)_{g_0} = A + g_0$.

Proof. If $L \in \mathfrak{M}_{f-g_0} \cap A^0$, then by Property A; $H[L, L(f) - \|f - g_0\|]$ supports the ball $B(f, \|f - g_0\|)$. For $\forall z \in (A)_{g_0}$ we have

$$L(z) = L(g_0) = L(f) - \|f - g_0\| .$$

Conversely, if H supports $B(f, r)$, by Property A we are ensured there exists a unique $L \in S_{E^*}$ such that $\forall y \in H$, $L(y) = L(f) - \|f - g_0\|$. Since H passes through $(A)_{g_0}$, $L(x)$ is a constant for all $x \in A$. As $0 \in A$, we have $L \in \mathfrak{M}_{f-g_0} \cap A^0$. Q.E.D.

Applying Theorem 12, Lemma 7(a) and Theorem 8, we obtain

THEOREM 13. *Let E be a normed linear space, G a subset of E, let $f \in E \backslash \operatorname{adh} G$ and $g_0 \in G$. Let the cone $C[g_0, G]$ contain a nonvoid linear subspace $\mathbf{M}_c[g_0, G]$.*

(a) *If $g_0 \in \mathfrak{L}_G(f)$, then:* (i) *there exists a hyperplane H which supports the ball $B(f, \|f - g_0\|)$ and passes through the translated linear subspace*

$$(\mathbf{M}_c[g_0, G])_{g_0} .$$

(b) *If the Local Kolmogoroff condition on G versus $\mathbf{M}_c[g_0, G]$ is sufficient, then $g_0 \in \mathfrak{L}_G(f)$ if and only if the preceding condition* (i) *holds.*

Obviously in Theorem 13, $\mathbf{M}_c[g_0, G]$ can be replaced by $\mathfrak{L}[a_0]$, if it is nonvoid. In Fig. 8 (i) of theorem 13 is illustrated.

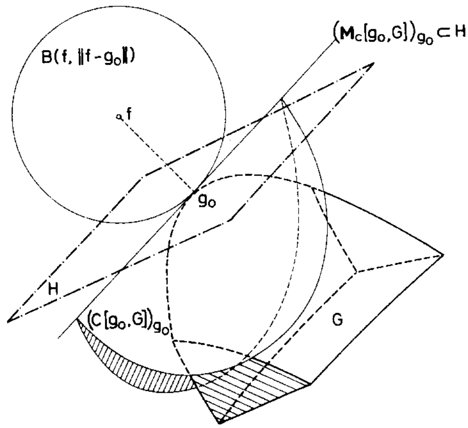

FIG. 8. Geometrical interpretation of the characterization theorem of type II.

5. Refinement of The Characterization Theorem of Type II

(5a) If the linear subspace $\mathbf{M}_c[g_0, G]$ is of finite dimension, the characterization theorems stated in Section 4 can be refined. This is based on Singer's following extension of Caratheodory's theorem:

THEOREM 14 [10, p. 169]. *Let E be a normed linear space and E_k a k-dimensional linear subspace of E. Let $L \in E^*$, $\| L|_{E_k} \| = 1$. There exist extremal points $L_1, ..., L_h$ of the unit ball B_{E^*}, where $h \leqslant k$ for a real E, $h \leqslant 2k - 1$ for a complex E, and positive scalars $\lambda_1, ..., \lambda_h$ with $\sum_{j=1}^{h} \lambda_j = 1$, such that*

$$L(g) = \sum_{j=1}^{h} \lambda_j L_j(g); \qquad \forall g \in E_k .$$

LEMMA 15. *Let E be a normed linear space, G a subset of E, let $f \in E \backslash \text{adh}\, G$ and $g_0 \in G$. Let $\mathbf{M}_c[g_0, G]$ be a nonvoid finite dimensional subspace in $C[g_0, G]$ of dimension d. If $g_0 \in \mathfrak{L}_G(f)$, then there exist linear functionals $L_1, ..., L_h \in \mathfrak{E}(B_{E^*})$ where $h \leqslant d + 1$ for a real E, $h \leqslant 2d + 1$ for a complex E and scalars $\lambda_1, ..., \lambda_h, \mu_1, ..., \mu_h$ such that the following equivalent conditions* (1)–(4) *are satisfied*:

(1) $\lambda_1, ..., \lambda_h$ are positive,

(i) $\sum_{j=1}^{h} \lambda_j = 1,$ (19)

(ii) $\sum_{j=1}^{h} \lambda_j L_j(k) = 0; \qquad \forall k \in \mathbf{M}_c[g_0, G],$ (20)

(iii) $\sum_{j=1}^{h} \lambda_j L_j(f - g_0) = \|f - g_0\|.$ (21)

(2) $\lambda_1 \cdots \lambda_h$ are positive, satisfy (19), (20) and

$$L_j(f - g_0) = \|f - g_0\|, \qquad j = 1 \cdots h \tag{22}$$

(3) Each μ_i is $\neq 0$, and

(i) $\sum_{j=1}^{h} |\mu_j| = 1,$ (23)

(ii) $\sum_{j=1}^{h} \mu_j L_j(k) = 0; \qquad \forall k \in \mathbf{M}_c[g_0, G],$ (24)

(iii) $\sum_{j=1}^{h} \mu_j L_j(f - g_0) = \|f - g_0\|.$ (25)

(4) Each μ_i is $\neq 0$, and we have (23), (24) and

$$L_j(f - g_0) = (\operatorname{sign} \mu_j) \cdot \|f - g_0\|; \quad j = 1,\dots, h \tag{26}$$

(where $\operatorname{sign} \alpha = \bar{\alpha}/|\alpha|$, for $\alpha \neq 0$).

Proof. By Lemma 7 we only need to prove equivalence between conditions (1) to (4) and the following

condition (0) : The set $\mathfrak{M}_{f-g_0} \cap \mathbf{M}_c{}^0[g_0, G]$ is nonvoid.

Defining $\mathbf{N} = \mathbf{M}_c[g_0, G] \oplus (f - g_0)$, one has $\| L|_{\mathbf{N}} \| = 1$ for

$$L \in \mathfrak{M}_{f-g_0} \cap \mathbf{M}_c{}^0[g_0, G].$$

By virtue of Theorem 14 there exist linear functionals $L_1, \dots, L_h \in \mathfrak{E}(B_{E^*})$ and numbers $\lambda_1, \dots, \lambda_h > 0$ such that we have (19) and

$$L(v) = \sum_{j=1}^{h} \lambda_j L_j(v); \qquad \forall v \in \mathbf{N}$$

which proves (0) $\to$ (1). For $j = 1,\dots, h$ we have

$$\operatorname{Re} L_j(f - g_0) = \|f - g_0\|$$

and

$$\|f - g_0\| \leqslant |L_j(f - g_0)| \leqslant \|f - g_0\|.$$

Since, if there were a $j_0 \in \{1, 2,..., h\}$ such that $\operatorname{Re} L_{j_0}(f - g_0) < \|f - g_0\|$, then

$$\|f - g_0\| = \operatorname{Re} \textstyle\sum_{j=1}^{h} \lambda_j L_j(f - g_0) < \sum_{j=1}^{h} \lambda_j \cdot \|f - g_0\| = \|f - g_0\|,$$

which is false. Consequently (1) → (2). Further, (2) → (4) → (3) is obvious. Finally (3) → (0) follows by taking $L = \sum_{j=1}^{h} \mu_j L_j$. Q.E.D.

THEOREM 16. *Let E be a normed linear space, G a subset of E, let $f \in E \backslash \operatorname{adh} G$ and $g_0 \in G$. Let $\mathbf{M}_c[g_0, G]$ be a nonvoid finite dimensional linear space in $C[g_0, C]$ with dim $\mathbf{M}_c[g_0, G] = d$. If the Local Kolmogoroff condition on G versus $\mathbf{M}_c[g_0, G]$ is sufficient, then $g_0 \in \mathfrak{L}_G(f)$ if and only if the equivalent conditions of Lemma* 15 *hold.*

COROLLARY 17. *Let E be a normed linear space, G a subset of E, let $f \in E \backslash \operatorname{adh} G$ and $g_0 \in G$. Let $\mathfrak{L}[a]$ be a nonvoid linear space for $\forall a \in P$. Let $\mathfrak{L}[a_0]$ be a finite dimensional linear space*:

$$\operatorname{span}\{l_k \mid k = 1 \cdots d[a_0]\} = \mathfrak{L}[a_0].$$

(a) *If $g_0 \in \mathfrak{L}_G(f)$ then there exist linear functionals $L_1, ..., L_h \in \mathfrak{E}(B_{E^*})$, where $1 \leqslant h \leqslant d[a_0] + 1$ if E is real, $1 \leqslant h \leqslant 2d[a_0] + 1$ if E is complex, and there exist numbers $\lambda_1, ..., \lambda_h, \mu_1, ..., \mu_h$ such that the following equivalent conditions* (1)–(4) *are satisfied*:

(1) $\lambda_1, ..., \lambda_h$ *are positive and satisfy* (19), (21), *and*

$$\sum_{j=1}^{h} \lambda_j L_j(l_k) = 0; \qquad k = 1,..., d[a_0]. \tag{27}$$

(2) $\lambda_1, ..., \lambda_h$ *are positive and satisfy* (19), (22) *and* (27).

(3) $\mu_1, ..., \mu_h$ *are nonzero, and satisfy* (23), (25) *and*

$$\sum_{j=1}^{h} \mu_j L_j(l_k) = 0; \qquad k = 1,..., d[a_0]. \tag{28}$$

(4) $\mu_1, ..., \mu_h$ *are nonzero and satisfy* (23), (26) *and* (28).

(b) *If, in addition, the Local Kolmogoroff condition on G versus $\mathbf{M}_c[g_0, G]$ is sufficient, then $g_0 \in \mathfrak{L}_G(f)$ if and only if the preceding conditions* (1)–(4) *hold.*

(5b) *Geometrical Interpretation*

Based on Theorem 12, a geometrical interpretation of Lemma 15, Theorem 16 and Corollary 17 can be obtained.

THEOREM 18. *Let E be a normed linear space, G a subset of E, let $f \in E \backslash \text{adh}\, G$ and $g_0 \in G$. Let $\mathbf{M}_c[g_0, G]$ be a nonvoid finite dimensional linear space in $C[g_0, G]$ with $d = \dim \mathbf{M}_c[g_0, G]$.*

(a) *If $g_0 \in \mathfrak{L}_G(f)$, then:* (i) *there exists a hyperplane H which is a convex combination of h extremal hyperplanes ($1 \leqslant h \leqslant d+1$ or $1 \leqslant h \leqslant 2d+1$ according to whether E is real or complex) each supporting the ball $B(f, \|f - g_0\|)$ at g_0; H passes through $(\mathbf{M}_c[g_0, G])_{g_0}$.*

(b) *If, in addition, the Local Kolmogoroff condition on G versus $\mathbf{M}_c[g_0, G]$ is sufficient, then $g_0 \in \mathfrak{L}_G(f)$ if and only if the preceding condition* (i) *holds.*

6. DUALITY RELATIONS

(6a) Basic duality relations pair up two extremal problems, the one in a given space, the other in the corresponding dual space. The following theorem links such dual problems.

THEOREM 19. *Let E be a normed linear space, G a subset of E, let $f \in E \backslash \text{adh}\, G$ and $g_0 \in G$.*

(a) *If* (i) $g_0 \in \mathfrak{L}_G(f)$,

(ii) $\mathbf{M}$ *is a nonvoid linear subspace in E ($\mathbf{M}$ not necessarily closed) such that $0 \in \mathfrak{L}_{\mathbf{M}}(f - g_0)$, then*

$$\|f - g_0\|_{\mathbf{M}^0} = \|f - g_0\|.$$

(b) *If* (i) *the Local Kolmogoroff condition on G is sufficient,*

(ii)

$$\|f - g_0\|_{C^0[g_0, G]} = \|f - g_0\|,$$

then $g_0 \in \mathfrak{L}_G(f)$.

Proof. (a) $\mathbf{M}^0 \subset E^*$ implies

$$\|f - g_0\|_{\mathbf{M}^0} \leqslant \|f - g_0\|.$$

By Lemma 7(a) there exists a linear functional $L \in \mathfrak{M}_{f-g_0} \cap \mathbf{M}^0$.

Moreover:

$$\|f - g_0\|_{\mathbf{M}^0} = \sup_{L' \in \mathbf{M}^0 \cap B_{E^*}} |L'(f - g_0)| \geqslant |L(f - g_0)| = \|f - g_0\|$$

(b) $$\|f - g_0\| = \sup_{L' \in C^0[g_0, G] \cap B_{E^*}} |L'(f - g_0)| \leqslant \|f - g_0 - k\|,$$

$$\forall k \in C[g_0, G].$$

Consequently we have $0 \in \mathfrak{L}_{C[g_0,G]}(f - g_0)$. By Lemma 7(b) and the assumption (i). on G, this is equivalent with $g_0 \in \mathfrak{L}_G(f)$. Q.E.D.

THEOREM 20. *Let E be a normed linear space and G a subset of E, let $f \in E \backslash \operatorname{adh} G$ and $g_0 \in G$. Let $\mathbf{M}_c[g_0, G]$ be any nonvoid linear subspace in $C[g_0, G]$.*

(a) *If $g_0 \in \mathfrak{L}_G(f)$, then*

$$\| f - g_0 \|_{\mathbf{M}_c{}^0[g_0,G]} = \| f - g_0 \|. \tag{29}$$

(b) *If the Local Kolmogoroff condition on G versus $\mathbf{M}_c[g_0, G]$ is sufficient, then $g_0 \in \mathfrak{L}_G(f)$ if and only if (29) holds.*

Proof. The proof is similar to that of Theorem 19, applying Theorem 8. In Theorem 20 one can replace $\mathbf{M}_c[g_0, G]$ by $\mathfrak{L}[a_0]$. In Fig. 9 the conclusion (29) of Theorem 20 is illustrated. To state complete duality relations we quote the following

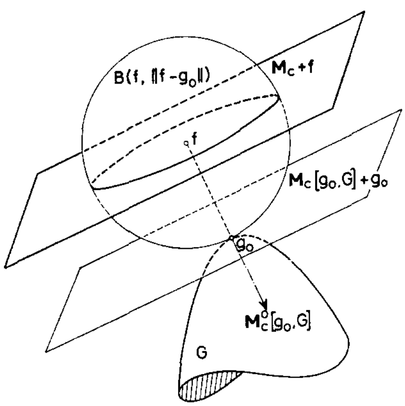

FIG. 9. If $g_0 \in \mathfrak{L}_G(f)$ then $\| f - g_0 \|_{\mathbf{M}_c{}^0[g_0,G]} = \| f - g_0 \|$.

THEOREM 21 [10, p. 21; Corollary 1, 2b]. *Let E^* be the conjugate space of a normed linear space E, and let Γ be a $\sigma(E^*, E)$-closed linear subspace of E^*. Let $L \in E^* \backslash \Gamma$ and $\gamma_0 \in \Gamma$. Then $\gamma_0 \in \mathfrak{L}_\Gamma(L)$ if and only if*

$$\| (L - \gamma) |_{{}^0\Gamma} \| = \| L - \gamma_0 \|.$$

(6b) *Relations*

(1a) $g_0 \in \mathfrak{L}_G(f)$ implies

$$\inf_{g \in G} \| f - g \| = \max_{L \in \mathbf{M}^0 \cap B_{E^*}} | L(f - g_0) |.$$

(1b) If the Local Kolmogoroff condition on G versus $\mathbf{M}_c[g_0, G]$ is sufficient, then $g_0 \in \mathfrak{L}_G(f)$ is equivalent with

$$\inf_{g\in G} \| f - g \| = \max_{L\in \mathbf{M}_c{}^0[g_0, G]\cap B_{E*}} | L(f - g_0)|. \tag{30}$$

(2) $\gamma_0 \in \mathfrak{L}_\Gamma(L)$ is equivalent with

$$\sup_{f\in {}^0\Gamma\cap B_E} |(L - \gamma_0)(f)| = \min_{\gamma\in\Gamma} \| L - \gamma \|. \tag{31}$$

(6c) *Geometrical Interpretation*

The relation (30) can be formulated as:

$$\rho(f, G) = \| f - g_0 \|_{\mathbf{M}_c{}^0[g_0, G]},$$

which indicates that the distance from f to the set G is given by the norm of $f - g_0$ on $\mathbf{M}_c{}^0[g_0, G]$. Similarly, relation (31) can be reformulated as:

$$\rho(L, \Gamma) = \|(L - \gamma_0)|_{{}^0\Gamma}\|$$

and expresses equality between the distance from L to the set Γ and the norm of the linear functional restricted to ${}^0\Gamma$.

7. Conclusion

Our main purpose was to present a unified approach to nonlinear approximation theory, extending some of Singer's results from the linear theory.

In general normed vector spaces, nonlinear approximation theory has already been provided with extensions of the Kolmogoroff condition [given by Brosowski in [3] and repeated here in Lemmas 1 and 2]. Based on these results and on the Hahn-Banach extension theorem we obtained a general characterization theorem (Theorem 8). If the functions are Gateaux (resp. Frechet) differentiable we have a characterization theorem which holds if some requirements concerning the subset $G \subset E$ are fulfilled (Theorem 11). This characterization theorem was further refined to obtain a more explicit formulation (Theorem 16). Throughout this paper geometrical interpretations were given for all characterization theorems. Finally, duality relations were given. As in linear approximation theory, they pair up two extremal problems, one in the normed space E, the other in the weak * dual.

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